PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On Hamilton's Numbers

J. J. Sylvester and James Hammond

Phil. Trans. R. Soc. Lond. A 1887 **178**, 285-312
doi: 10.1098/rsta.1887.0010

X. *On Hamilton's Numbers.*

By J. J. SYLVESTER, *D.C.L., F.R.S.*,
Savilian Professor of Geometry in the University of Oxford,
and JAMES HAMMOND, *M.A. Cant.*

Received June 11,—Read June 16, 1887.

INTRODUCTION.

IN the year 1786 ERLAND SAMUEL BRING, Professor at the University of Lund in Sweden, showed how by an extension of the method of TSCHIRNHAUSEN it was possible to deprive the general algebraical equation of the 5th degree of three of its terms without solving an equation higher than the 3rd degree. By a well-understood, however singular, academical fiction, this discovery was ascribed by him to one of his own pupils, a certain SVEN GUSTAF SOMMELIUS, and embodied in a thesis humbly submitted to himself for approval by that pupil, as a preliminary to his obtaining his degree of Doctor of Philosophy in the University.* The process for effecting this reduction seems to have been overlooked or forgotten, and was subsequently re-discovered many years later by Mr. JERRARD. In a report contained in the 'Proceedings of the British Association' for 1836, Sir WILLIAM HAMILTON showed that Mr. JERRARD was mistaken in supposing that the method was adequate to taking away more than three terms of the equation of the 5th degree, but supplemented this somewhat unnecessary refutation of a result, known *à priori* to be impossible, by an extremely valuable discussion of a question raised by Mr. JERRARD as to the number of variables required in order that any system of equations of given degrees in those variables shall admit of being satisfied without solving any equation of a degree higher than the highest of the given degrees.

In the year 1886 the senior author of this memoir showed in a paper in KRONECKER'S (better known as CRELLE'S) 'Journal' that the trinomial equation of

* BRING'S "Reduction of the Quintic Equation" was republished by the Rev. ROBERT HARLEY, F.R.S., in the 'Quarterly Journal of Pure and Applied Mathematics,' vol. 6, 1864, p. 45. The full title of the LUND Thesis, as given by Mr. HARLEY (see 'Quart. Journ. Math.,' pp. 44, 45) is as follows: "B. cum D. Meletemata quaedam mathematica circa transformationem aequationum algebraicarum, quae consent. Ampliss. Facult. Philos. in Regia Academia Carolina Praeside D. ERLAND SAM. BRING, Hist. Profess. Reg. & Ord. publico Eruditorum Examini modeste subjicit SVEN GUSTAF SOMMELIUS, Stipendiarius Regius & Palmcrentzianus Lundensis. Die XIV Decemb., MDCCLXXXVI, L.H.Q.S.—Lundae, typis Berlingianis."

30.9.87

the 5th degree, upon which by BRING's method the general equation of that degree can be made to depend, has necessarily imaginary coefficients except in the case where four of the roots of the original equation are imaginary, and also pointed out a method of obtaining the absolute minimum degree M of an equation from which any given number of specified terms can be taken away subject to the condition of not having to solve any equation of a degree higher than M.* The numbers furnished by HAMILTON's method, it is to be observed, are not minima unless a more stringent condition than this is substituted, viz., that the system of equations which have to be resolved in order to take away the proposed terms shall be the simplest possible, *i.e.*, of the lowest possible weight and not merely of the lowest order; in the memoir in 'CRELLE,' above referred to, he has explained in what sense the words weight and order are here employed. He has given the name of HAMILTON's Numbers to these relative minima (minima, *i.e.*, in regard to weight) for the case where the terms to be taken away from the equation occupy consecutive places in it, beginning with the second.

Mr. JAMES HAMMOND has quite recently discovered by the method of generating functions a very simple formula of reduction, or scale of relation, whereby any one of these numbers may be expressed in terms of those that precede it: his investigation, which constitutes its most valuable portion, will be found in the second section of this paper. The principal results obtained by its senior author, consequential in great measure to Mr. HAMMOND's remarkable and unexpected discovery, refer to the proof of a theorem left undemonstrated in the memoir in 'CRELLE' above referred to, and the establishment of certain other asymptotic laws to which HAMILTON's Numbers and their differences are subject, by a mixed kind of reasoning, in the main apodictic, but in part also founded on observation.† It thus became necessary to calculate out the 10th Hamiltonian Number, which contains 43 places of figures. The highest number calculated by HAMILTON (the 6th) was the number 923, which comes third in order after 5 (the BRING Number), 11 and 47 being the two intervening numbers. It is to be hoped that some one will be found willing to undertake the labour (considerable, but not overwhelming) of calculating some further numbers in the scale.

The theory has been "a plant of slow growth." The LUND Thesis of December,

* For instance, an equation of not lower than the 905th degree may be transformed into another of that degree, in which the 2nd, 3rd, 4th, 5th, 6th, 7th, terms are all wanting, by means of the successive solution of a ramificatory system of equations, of no one of which the degree exceeds 6, whereas by the Jerrard-Hamiltonian method this transformation could not be effected for the general equation of degree lower than the 6th Hamiltonian Number, viz., 923. So for the analogous removal of 5 consecutive terms the inferior limit of degree of the equation to be transformed would be 47 by the one method, but 44 (the lowest possible) by the other. In the case of 4 consecutive terms HAMILTON could not avoid being aware that 11, the 4th number which I have named after him, might be replaced by 10, as the lowest possible inferior limit of the equation to be transformed.

† In the 3rd section, communicated to the Society after the 1st and 2nd had gone to press, the empirical element is entirely eliminated, and the results reduced to apodictic certainty.

1786 (a matter of a couple of pages), HAMILTON'S Report of 1836, with the tract of Mr. JERRARD therein referred to, and the memoir in 'CRELLE' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date.

§ 1. *On the Asymptotic Laws of the Numbers of* HAMILTON *and their Differences.*

Consider the following Table :—

1	0	0	0	0	0	0	0	....
	1	1	1	1	1	1	1	....
		2	3	4	5	6	7	....
			6	15	29	49	76	....
				36	210	804	2449	....
					876	24570	401134	....
						408696	246382080	....
							83762796636	....
								....
								...

Any line of figures, say $p, q, r, s, t, \ldots \theta$, in the Table being given, to form the subsequent line $q_1, r_1, s_1, t_1, \ldots \theta_1$, we write

$$q_1 = \frac{p(p+1)}{1.2} + q.$$

$$r_1 = \frac{p(p+1)(2p+1)}{1.2.3} + pq + r.$$

$$s_1 = \frac{p(p+1)(p+2)(3p+1)}{1.2.3.4} + \frac{p(p+1)}{1.2} q + pr + s.$$

$$t_1 = \frac{p(p+1)(p+2)(p+3)(4p+1)}{1.2.3.4.5} + \frac{p(p+1)(p+2)}{1.2.3} q + \frac{p(p+1)}{1.2} r + ps + t.$$

. . . .

$$\theta_1 = \frac{p(p+1)\ldots(p+i-1)(ip+1)}{1\,2.3\ldots(i+1)} + \frac{p(p+1)\ldots(p+i-2)}{1.2.3\ldots(i-1)} q + \ldots + \theta.$$

If we call the n^{th} term of the m^{th} line $[m, n]$, the general law of deduction may be expressed by the formula

$$[m+1, n] = -\mathrm{B}_{n+1}([m, 1] - 1) + \sum_{i=0}^{i=n} [m, n+1-i]\,\mathrm{B}_i[m, 1],$$

where $\mathrm{B}_i k$ means the coefficient of z^i in $(1-z)^{-k}$.

The negative term $-B_{n+1}([m, 1] - 1)$, it may be noticed, arises from decomposing the first term of $[m + 1, n]$, as given by the original formulæ, into two parts, of which it is one.

Thus, *ex. gr.*,

$$\frac{p(p+1)(p+2)(p+3)(4p+1)}{1.2.3.4.5}$$

is changed into

$$-\frac{(p-1)p(p+1)(p+2)(p+3)}{1.2.3.4.5} + \frac{p(p+1)(p+2)(p+3)}{1.2.3.4}p.$$

The numbers in the hypothenuse of this infinite triangle, viz.,

1, 1, 2, 6, 36, 876, 408696, 83762796636, 3508125906207095591916,
6153473687096578758445014683368786661634996, ... ,

are what I call the Hamiltonian Differences, or Hypothenusal Numbers* ; and their continued sums augmented by unity, viz.,

2, 3, 5, 11, 47, 923, 409619, 83763206255, 3508125906290858798171,
6153473687096578758448522809275077520433167, ... ,

are what I call the Hamiltonian Numbers. The two latter of these have been calculated by means of Mr. HAMMOND's formula, presently to be mentioned, and the corresponding Hypothenusal Numbers deduced from them by simple subtraction. Their connection with the theory of the TSCHIRNHAUSEN Transformation will be found fully explained in my memoir on the subject in vol. 100 of 'CRELLE.' My present object is to speak of the numbers as they stand, without reference to their origin or application.†

* The other numbers of the "triangle," whose properties it may be some day desirable to investigate, may be termed co-hypothenusal numbers of *order* measured by their horizontal distance from the hypothenuse—their vertical distance below the top line denoting their *rank*. In the sequel the development is given of the half of a hypothenusal number (of the first order) in a descending series of powers (with fractional indices) of the half of its antecedent, the coefficients in the principal part of such series being (not, as might have been the case, functions of the rank, but) absolute constants. These may be termed the hypothenusal constants. The values of the first four of them are shown to be 1, $\frac{4}{3}$, $\frac{11}{18}$, $\frac{10}{81}$.

† The reader will be disappointed who seeks in HAMILTON's Report any systematic deduction of the numbers which I have called after his name. He treats therein the more general question of finding the number of letters sufficient for satisfying any system of equations of given degrees by means of a certain prescribed uniform process whereby the necessity is obviated of solving any equation of a higher degree than the highest one of the given equations, and among, and mixed up with, other examples considers systems of equations of degrees 1, 2, 3; 1, 2, 3, 4; 1, 2, 3, 4, 5; 1, 2, 3, 4, 5, 6; for which the minimum numbers of letters required to make such process possible (when the equations are homogeneous) are 5, 11, 47, 923, respectively. Accordingly he has no occasion to employ the infinitely developable Triangle which gives unity and cohesion to the problem which deals with an indefinite number of equations of all consecutive degrees from 1 upwards. This triangle, which plays an impor-

The question arises as to whether it is possible to deduce the Hamiltonian Differences, or to deduce the Hamiltonian Numbers, directly in a continued chain from one another without the use of any intermediate numbers. Mr. JAMES HAMMOND has shown that it is possible, and has made the remarkable discovery that it is the Numbers of Hamilton, and not the Hypothenusal Numbers, which are subject to a very simple scale of relation. These being found, of course the Differences become known. This is contrary to what one would have expected. *A priori*, one would have anticipated that the determination of the Hypothenusal Numbers would have preceded that of their sums.

I leave Mr. HAMMOND to give his own account of his mode of obtaining the wonderful formula of reduction, which, by a slight modification, I find, may be expressed as follows :—Using E_i to denote the $(i+1)^{\text{th}}$ Hamiltonian Number augmented by unity, so that $E_0 = 3$, $E_1 = 4$, $E_2 = 6$, $E_3 = 12$, $E_4 = 48$, . . . ; and $\beta_i m$ to signify the coefficient of t^i in $(1+t)^m$; then, for any value of i greater than unity,

$$\beta_0 E_i - \beta_1 E_{i-1} + \beta_2 E_{i-2} - \beta_3 E_{i-3} + \ldots + (-)^i \beta_i E_0 = 0.$$

tant part in the systematic treatment of the problem, first appears in my memoir on the subject in the 100th volume of 'CRELLE.'

It is proper also again to notice that what I call the Numbers of HAMILTON (at all events those subsequent to the number 5) are not the smallest numbers requisite for fulfilling the condition above specified. Smaller numbers will serve to satisfy that condition taken alone; but when such smaller numbers are substituted for HAMILTON'S the resolving equations will be less simple, inasmuch as they will contain a greater number of equations of the higher degrees than when the larger Hamiltonian numbers are employed. This distinction will be found fully explained in the memoir cited, and the smallest numbers substitutable for HAMILTON'S are there actually determined for r equations of degrees extending from 1 to r for all values of r up to 8 inclusive.

I have added nothing (for there is nothing to be added) to the fundamental formula of HAMILTON expressed by the equation

$$[\lambda, \mu, \nu, \ldots \pi] = 1 + [\lambda - 1, \lambda + \mu, \lambda + \mu + \nu, \ldots, \lambda + \mu + \nu + \ldots + \pi],$$

where, supposing the letters $\lambda, \mu, \nu, \ldots \pi$, to be i in number, $[\lambda, \mu, \nu, \ldots \pi]$ means the number of letters required in order that it may be possible to satisfy, according to the process employed by HAMILTON (in conformity with a certain stipulation of JERRARD), a system of λ equations of degree i, μ equations of degree $i-1$, ν equations of degree $i-2$, . . . , π equations of the degree 1, without solving any single equation of a degree higher than i. This formula, applied λ times successively, will have the effect of abolishing λ and causing $[\lambda, \mu, \nu, \ldots \pi]$ to depend on $[\mu', \nu', \ldots \pi']$, where $\mu', \nu', \ldots \pi'$ are connected with $\lambda, \mu, \nu, \ldots \pi$ by means of the formulæ given at the commencement of the present paper, but where instead of the letters $\lambda, \mu, \nu, \ldots$ I have used the letters $p, q, r, \ldots$

It is presumable that the *reduced* Hamiltonian numbers would be found much less amenable to algebraical treatment than the Hamiltonian numbers proper; for numerical equalities and inequalities have to be taken account of, in determining them, which have no place in the determination of the latter numbers. HAMILTON, as already stated, expressly alludes to the reduction of 11 to 10, but with that exception has avoided the general question of finding the *absolutely* lowest number of letters required in order that a system of equations (expressed in terms of those letters) of given degrees may admit of being satisfied without the necessity arising to solve any equation of a higher degree than the highest of the given ones.

Or in other words, writing $\beta_0 E_i = 1$, $\beta_1 E_{i-1} = E_{i-1}$, and replacing $i - 1$ by i,

$$E_i = 1 + \beta_2 E_{i-1} - \beta_3 E_{i-2} + \ldots + (-)^{i+1} \beta_{i+1} E_0$$

for all values of i greater than zero.

This is eminently a practical formula, as all the numerical calculations made use of to obtain any E are available for finding the E which follows. Dispensing with the symbol β, we may deduce all the values of E successively from those that go before by means of the *equivalence*

$$S = (1-t)^{E_0} + t(1-t)^{E_1} + t^2(1-t)^{E_2} + \ldots \equiv 1 - 2t,$$

which, by equating the powers of t on the two sides of the equivalence, gives

$$E_0 = 3,$$

$$E_1 = 1 + \frac{3.2}{1.2} = 4,$$

$$E_2 = 1 + \frac{4.3}{1.2} - \frac{3.2.1}{1.2.3} = 6,$$

$$E_3 = 1 + \frac{6.5}{1.2} - \frac{4.3.2}{1.2.3} + \frac{3.2.1.0}{1.2.3.4} = 12,$$

and so on.

I use the term equivalence and its symbol in order to convey the necessary caution that the relation indicated is not one of quantitative equality; for, although the series on the left-hand side of the symbol converges for all positive values of t less than 2, it is never equal to the expression on the right-hand side except when $t = 0$. Thus, *e.g.*, when t is unity the two terms of the equivalence are 0 and -1, and when $t = \frac{1}{2}$ they are

$$2^{-E_0} + 2^{-E_1-1} + 2^{-E_2-2} + \ldots \text{ and } 0, \text{ respectively;}$$

and for all values of t within the limits of convergence the value of the left-hand side is in excess of the value of the right-hand side of the equivalence by a finite quantity which decreases continuously as t decreases from 2 to 0, and which vanishes when $t = 0$.*

In a word, the *generating equation* is not an equation in the usual sense of the term. Conceiving each term of the series S to be expanded in ascending powers of t, and like powers of t to be placed in columns under and above each other, the double

* Of the truth of the statement that the excess never changes sign, and continually decreases, I have scarcely a doubt, but it requires proof. Mr. HAMMOND remarks that

$$(1-t)^{E_0} + t(1-t)^{E_1} + t^2(1-t)^{E_2} + \ldots + t^n(1-t)^{E_n} = (1-2t) + t^2(1-t)^{E_n-2} F_n(t) - t^{n+1}(1-t)^{E_n-1},$$

where $F_n(t)$ is positive for all positive values of t. Probably a proof of the point in question might be deduced from this expression, but I have not thought it necessary to investigate the matter.

sum may be taken as a vertical sum of line-sums or as a horizontal sum of column-sums, and, although for licit values of t each sum has a finite value, the two finite values are not identical, just as a double definite integral may undergo a change of value when the order of its integrations is reversed.*

I noticed at p. 478 of the 100th volume of 'CRELLE' that the value of any Hamiltonian Difference divided by the square of the preceding one was always greater than $\frac{1}{2}$, and stated as morally certain, but "awaiting exact proof," that this ratio ultimately becomes $\frac{1}{2}$. By aid of Mr. HAMMOND's formula for the numbers, I shall now be able to supply this proof, and at the same time to show that the ratio of a Hamiltonian *Number* to the square of its antecedent (which, of course, converges to the same asymptotic value $\frac{1}{2}$) is always *less* than that limit.†

We must in the first place prove that in the series

$$\beta_2 E_{i-1} - \beta_3 E_{i-2} + \beta_4 E_{i-3} - \beta_5 E_{i-4} + \ldots$$

the absolute value of each term is greater than that of the one which follows it.

In proving this, I shall avail myself of the property of the Hypothenusal Numbers disclosed in the process of forming the triangle given at the outset of the memoir, viz., that $E_i - E_{i-1}$ is greater than $(E_{i-1} - E_{i-2})^2/2$.

Let us suppose that the law to be established holds good for a certain value of i. For the sake of brevity, I denote E_i, E_{i-1}, E_{i-2}, E_{i-3}, ... by N, P, Q, R, ...

We have then

$$P - 1 = \frac{Q(Q-1)}{2} - \frac{R(R-1)(R-2)}{2.3} + \frac{S(S-1)(S-2)(S-3)}{2.3.4} - \ldots$$

$$N - 1 = \frac{P(P-1)}{2} - \frac{Q(Q-1)(Q-2)}{2.3} + \frac{R(R-1)(R-2)(R-3)}{2.3.4} - \frac{S(S-1)(S-2)(S-3)(S-4)}{2.3.4.5} + \ldots$$

* Professor CAYLEY has brought under my notice a not altogether dissimilar, but perhaps less striking, phenomenon, pointed out by CAUCHY, that, although the series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$$

is convergent, its square

$$u_0^2 + (2u_0u_1) + (2u_0u_2 + u_1^2) + \ldots,$$

i.e.,

$$1 - \sqrt{2} + \left(\frac{2}{\sqrt{3}} + \tfrac{1}{2}\right) - \ldots$$

is divergent.

† The fortunate circumstance of the two ratios in question being always respectively less and greater than the common asymptotic value of each of them enables us to find the value of the constant in the expression c^{2^x}, which is asymptotically equivalent to the half of the x^{th} Hamiltonian or Hypothenusal Number by a method exactly analogous to that of exhaustions for finding the Archimedean constant correct to any required number of decimal places. See end of this section (pp. 298, 299).

If, then, the law to be proved is true for all the consecutive terms of the upper series it will obviously be true for the second series, *abstraction being made of its first term*, provided that no antecedent is less than its consequent in the series

$$\frac{Q-2}{3}, \quad \frac{R-3}{4}, \quad \frac{S-4}{5}, \ldots,$$

which is true *à fortiori* if

$$\frac{Q}{3}, \quad \frac{R}{4}, \quad \frac{S}{5}, \ldots$$

continually decrease, as is obviously the case, inasmuch as

$$Q, \quad R, \quad S, \ldots$$

form a descending series.

In order, then, to establish the necessary chain of induction, it only remains to show that

$$3P(P-1) - Q(Q-1)(Q-2)$$

is positive.

Now

$$(P-Q) - \frac{(Q-R)^2}{2} \left[\text{and } \textit{à fortiori } P - \frac{(Q-R)^2}{2}\right]$$

is positive for a reason previously given.

And, if in the series 3, 4, 6, 12, 48, 924, . . . we make exclusion of the first three terms, we have always

$$R = \text{or} < \frac{Q}{4},$$

and consequently

$$P > \frac{9Q^2}{32}.*$$

And, since under the same condition $(P-1)/(Q-1) > 4$, $3P(P-1) - Q^2(Q-1)$, and *à fortiori* $3P(P-1) - Q(Q-1)(Q-2)$, is positive if $12P - Q^2$ is positive, which is the case, since $P > 9Q^2/32$.

Hence, since the theorem to be proved is true for the several series

$$(1) \quad \frac{4.3}{1.2} - \frac{3.2.1}{1.2.3},$$

$$(2) \quad \frac{6.5}{1.2} - \frac{4.3.2}{1.2.3},$$

$$(3) \quad \frac{12.11}{1.2} - \frac{6.5.4}{1.2.3} + \frac{4.3.2.1}{1.2.3.4},$$

$$(4) \quad \frac{48.47}{1.2} - \frac{12.11.10}{1.2.3} + \frac{6.5.4.3}{1.2.3.4},$$

* The proof that the ratio of each term of the series 4, 6, 12, 48, 924, . . . to its antecedent continually increases is too easy and too tedious to be worth setting forth in the text.

it will be true universally; for in all the succeeding series the term we have called R will be higher than the term 6 in the scale 3, 4, 6, 12, 48, . . .

Hence

$$P - 1 = \text{or} < \tfrac{1}{2}(Q^2 - Q).$$

For the initial values of Q, P, (viz., 3, 4,)

$$P - 1 = \tfrac{1}{2}(Q^2 - Q).$$

[When P represents any term beyond the first it is very easy to prove, but too tedious to set out the proof, that the sum of all the terms after the first in the series equated to $P - 1$ will be less than -2; so that, except in the case stated, $P < \frac{1}{2}(Q^2 - Q)$].

For the series 12, 48, 924, . . . we have seen that $P > 9Q^2/32$.

Hence, for the series 48, 924, . . . ,

$$Q > \frac{9R^2}{32} \quad \text{or} \quad R < \sqrt{\frac{32Q}{9}}.$$

But

$$P > \frac{Q^2 - Q}{2} - \frac{R(R-1)(R-2)}{6},$$

$$> \frac{Q^2 - Q}{2} - \frac{R^3}{6}.$$

Hence

$$P > \frac{Q^2 - Q}{2} - \tfrac{64\sqrt{2}}{81} Q^{\frac{3}{2}}, \quad \text{and} \quad P < \frac{Q^2 - Q}{2}.$$

Hence, when P, Q, are at an infinite distance from the origin,

$$\frac{P}{Q^2} = \tfrac{1}{2}.$$

Hence, also,

$$\frac{P - Q}{(Q - R)^2} \text{ ultimately} = \frac{P}{Q^2} = \tfrac{1}{2},$$

which proves the theorem left over for "exact proof" in the memoir referred to.

It is convenient to deal with the halves of the *sharpened** Numbers of HAMILTON, which may be called the reduced Hamiltonian Numbers, and denoted by h with a subscript, or, when required, by $p, q, r, \ldots$ (the halves of P, Q, R, . . . respectively).

We have then

$$2p < \frac{4q^2 - 2q}{2},$$

* Numbers increased by unity may conveniently be denominated sharpened numbers, and numbers diminished by unity flattened numbers.

or

$$p < q^2 - \frac{q}{2},$$

$$p > q^2 - \frac{q}{2} - \tfrac{128}{81} q^{\frac{3}{2}}.$$

We may find a closer superior limit to p in terms of q as follows—

$$\mathrm{P} - 1 = \text{or} < \frac{\mathrm{Q}^2 - \mathrm{Q}}{2} - \frac{\mathrm{R}(\mathrm{R}-1)(\mathrm{R}-2)}{6} + \frac{\mathrm{S}(\mathrm{S}-1)(\mathrm{S}-2)(\mathrm{S}-3)}{24},$$

in which inequality it may be shown by inspection up to a certain point, and after that by demonstration, the tedium of writing out or reading which I spare my readers and myself, that P may be substituted for its flattened value P − 1.

We have then

$$\mathrm{P} < \frac{\mathrm{Q}^2 - \mathrm{Q}}{2} - \frac{\mathrm{R}^3 - 3\mathrm{R}^2}{6} + \frac{\mathrm{S}^4}{24}.$$

Let us suppose that S, R, are not lower in the scale of the E's than 12, 48, respectively; so that P is not lower than E_6, which is 409620.

Then, as we have previously shown,

$$\mathrm{Q}^2 < \tfrac{32}{9}\mathrm{P}, \qquad \mathrm{R}^2 < \tfrac{32}{9}\mathrm{Q}, \qquad \mathrm{S}^2 < \tfrac{32}{9}\mathrm{R}.$$

Moreover, we have

$$\mathrm{P} < \tfrac{1}{2}(\mathrm{Q}^2 - \mathrm{Q}), \text{ whence it follows that } \mathrm{Q}^2 > 2\mathrm{P} + \mathrm{Q},$$

and, *à fortiori*,

$$\mathrm{Q}^2 > 2\mathrm{P}.$$

Similarly

$$\mathrm{R}^2 > 2\mathrm{Q},$$

and

$$\mathrm{S}^2 > 2\mathrm{R}.$$

Now

$$\mathrm{P} < \frac{\mathrm{Q}^2 - \mathrm{Q}}{2} - \frac{\mathrm{R}^3}{6} + \frac{\mathrm{R}^2}{2} + \frac{\mathrm{S}^4}{24}$$

$$< \frac{\mathrm{Q}^2 - \mathrm{Q}}{2} - \tfrac{1}{6}(2\mathrm{Q})^{\frac{3}{2}} + \tfrac{1}{2}(\tfrac{32}{9}\mathrm{Q}) + \tfrac{1}{24}(\tfrac{32}{9}\mathrm{R})^2$$

$$< \frac{\mathrm{Q}^2}{2} - \tfrac{\sqrt{2}}{3}\mathrm{Q}^{\frac{3}{2}} + \tfrac{23}{18}\mathrm{Q} + \tfrac{1}{24}(\tfrac{32}{9})^3\mathrm{Q},$$

i.e.,

$$\mathrm{P} < \tfrac{1}{2}\mathrm{Q}^2 - \tfrac{\sqrt{2}}{3}\mathrm{Q}^{\frac{3}{2}} + \tfrac{13781}{4374}\mathrm{Q}.$$

This result, expressed in terms of the reduced numbers p, q, takes the form

$$p < q^2 - \tfrac{2}{3}q^{\frac{3}{2}} + \tfrac{13781}{4374}q,$$

and we have previously shown that

$$p > q^2 - \tfrac{128}{81} q^{\frac{3}{2}} - \frac{q}{2},$$

at all events when P is not lower in the scale than E_6.

The fraction $\frac{128}{81}$ arises from our having substituted for R^3 the inferior value $(\frac{32}{9} Q)^{\frac{3}{2}}$; but, the higher we advance P in the scale, the nearer R approaches to 2Q, and is ultimately in a ratio of equality with it. But, if we had written $(2Q)^{\frac{3}{2}}$ for R^3, the coefficient, which now stands at $-\frac{128}{81}$, would have been $-\frac{2}{3}$. In like manner, as P and Q are travelled on in the scale, R^2 and S^4 become indefinitely near to 2Q and $(2R)^2$, *i.e.*, 8Q, so that the coefficient of Q in the superior limit approximates indefinitely near to

$$-\tfrac{1}{2} + 1 + \tfrac{1}{3}, \text{ i.e., } \tfrac{5}{6},$$

and the two limits of p which have been obtained become

$$q^2 - \tfrac{2}{3} q^{\frac{3}{2}} + (\tfrac{5}{6} + \epsilon) q,$$
$$q^2 - (\tfrac{2}{3} + \eta) q^{\frac{3}{2}} - \tfrac{1}{2} q,$$

where ultimately ϵ and η are infinitesimals.*

Hence it follows that the ultimate value of

$$(p - q^2) \div q^{\frac{3}{2}} \text{ is } -\tfrac{2}{3},$$

i.e.,

$$\frac{2E_i - E^2_{i-1}}{E^{\frac{3}{2}}_{i-1}} = -\sqrt{\tfrac{8}{9}} \text{ when } i = \infty.$$

Let λ, μ, ν, . . . represent the halves of the Hypothenusal Numbers in the triangle given at the commencement of the paper, *i.e.*, the differences of the numbers which we have called p, q, r, . . .

Since

$$p = q^2 - \tfrac{2}{3} q^{\frac{3}{2}} \quad \text{and} \quad q = r^2 - \tfrac{2}{3} r^{\frac{3}{2}},$$
$$p - q = q^2 - \tfrac{2}{3} q^{\frac{3}{2}} - q, \quad \text{and} \quad q - r = r^2 - \tfrac{2}{3} r^{\frac{3}{2}} - r.$$

Obviously, therefore, as a first approximation when λ, μ, are very advanced terms in the hypothenuse,

$$\lambda = \mu^2.$$

Let us write

$$\lambda = \mu^2 + \kappa\mu^a$$

for a second approximation.

* As a matter of fact, it will be found that, as soon as q and p attain the values 6, 24, $q^2 - \frac{2}{3} q^{\frac{3}{2}}$ may be taken as a superior limit. It may be noticed also, to prevent a wrong inference being drawn from the above expressions, that, as will hereafter appear, η is an infinitesimal of the order $1/q^{\frac{1}{4}}$, when q is infinite.

Then

$$q^2 - \tfrac{2}{3} q^{\frac{3}{2}} - q = (r^2 - \tfrac{2}{3} r^{\frac{3}{2}} - r)^2 + \kappa (r^2 - \tfrac{2}{3} r^{\frac{3}{2}} - r)^{\alpha},$$

or, *neglecting* terms of lower dimensions than r^3,

$$(r^2 - \tfrac{2}{3} r^{\frac{3}{2}})^2 - \tfrac{2}{3} r^3 \left(1 - \frac{1}{r^{\frac{1}{2}}} + \frac{1}{6r} - \dots\right) = (r^2 - \tfrac{2}{3} r^{\frac{3}{2}} - r)^2 + \kappa r^{2\alpha}.$$

Therefore

$$-\tfrac{2}{3} r^3 = -2r^3 + \kappa r^{2\alpha}.$$

Consequently

$$\alpha = \tfrac{3}{2} \quad \text{and} \quad \kappa = \tfrac{4}{3}.$$

Thus, then, for the consecutive Hypothenusal Numbers λ, μ,

$$\lambda = \mu^2 + \tfrac{4}{3} \mu^{\frac{3}{2}} + \dots$$

Let

$$\lambda = \mu^2 + \tfrac{4}{3} \mu^{\frac{3}{2}} + \theta \mu,$$

or say

$$\eta_{x+1} = \eta_x^2 + \tfrac{4}{3} \eta_x^{\frac{3}{2}} + \rho_x \eta_x,$$

where η_x is the x^{th} term in the series $\frac{1}{2}$, 1, 3, 18, . . .

The successive values of ρ_x and their differences are given in the annexed Table.

x	η_x	ρ_x	$\Delta\rho_x$
1	·5	·55719096	
2	1	·66666666	+·10947570
3	3	·69059893	+·02393227
4	18	·67647909	—·01411984
5	438	·64334761	—·03313148
6	204348	·61769722	—·02565039
7	41881398318	·61139243	—·00630479
8	1754062953103547795958	·61111171	—·00028072

The decimal figures following those given in ρ_8, required for ulterior purposes, being 5795.

An examination of the column of differences for $x = 5, 6, 7, 8$, shows that the ratios of each to the rest go on decreasing somewhat faster than their squares: this makes it almost certain that $\rho_8 - \rho_9$ will be between the 400th and 500th part of ·000280, and that accordingly the value of ρ_9 will be ·6111111, &c. I believe it is beyond all moral doubt that the ultimate value of ρ is exactly $\frac{11}{18}$; and, indeed, it was the conviction I entertained of this being its true value, when I had calculated ρ_7, that led me to undertake the very considerable labour of ascertaining the 10th Hamiltonian Number in order to deduce from it the value of ρ_8. This being taken for granted,* we may proceed to ascertain a further term in the asymptotic value of η_{x+1} expressed as a function of η_x.

For, calling

$$\rho_x - \tfrac{11}{18} = \delta_x \quad \text{and} \quad \sqrt[4]{\eta_x} = q_x,$$

we have

$$\delta_6 = \cdot 00658611,$$
$$\delta_7 = \cdot 00028132,$$
$$\delta_8 = \cdot 0000006047,$$
$$\left.\begin{aligned} q_6 &= 21, \\ q_7 &= 452, \\ q_8 &= 204649, \end{aligned}\right\} \text{neglecting decimals.}$$

Thus

$$(\delta q)_6 = \cdot 1383,$$
$$(\delta q)_7 = \cdot 1272,$$
$$(\delta q)_8 = \cdot 12375.$$

The value of

$$(\delta q)_6 - (\delta q)_7 \text{ being } \cdot 0111,$$

and of

$$(\delta q)_7 - (\delta q)_8 \qquad \cdot 0035,$$

we may feel tolerably certain, from the Law of Squares, that $(\delta q)_8 - (\delta q)_9$ will be somewhere in the neighbourhood of the tenth part of ·0035, and accordingly that $(\delta q)_9$ is about ·1234, so that the probable value of $(\delta q)_\infty$ is ·1234 . . .

Thus we have found

$$\eta_{x+1} = \eta_x^2 + \tfrac{4}{3}\, \eta_x^{\frac{3}{2}} + \tfrac{11}{18}\, \eta_x + [\ \ ]\, \eta_x^{\frac{3}{4}} + \ldots,$$

the only moral doubt being as to the degree of closeness of propinquity of the coefficient of $\eta_x^{\frac{3}{4}}$ to the decimal ·1234 . . . †

For the benefit of those who may wish to carry on the work, I give the following numerical results which have been employed in the preceding arithmetical determinations :—

* It is reduced to *certainty* in the supplemental 3rd section.

† The exact value of the coefficient of $\eta_x^{\frac{3}{4}}$, left blank in the text, is proved in section 3 to be $\frac{10}{81}$, *i.e.*, the recurring decimal $\cdot\dot{1}2345679\dot{0}$.

$$\frac{E_8(E_8-1)}{1.2} = 6153473687194529702895764001115884685871706$$

$$\frac{E_7(E_7-1)(E_7-2)}{1.2.3} = 9795094444841421613760720063 7520$$

$$\frac{E_6(E_6-1)(E_6-2)(E_6-3)}{1.2.3.4} = 117302430235229583 8445$$

$$\frac{E_5(E_5-1)(E_5-2)(E_5-3)(E_5-4)}{1.2.3.4.5} = 5552272910184$$

$$\frac{E_4(E_4-1)(E_4-2)(E_4-3)(E_4-4)(E_4-5)}{1.2.3.4.5.6} = 12271512$$

$$\frac{E_3(E_3-1)(E_3-2)(E_3-3)(E_3-4)(E_3-5)(E_3-6)}{1.2.3.4.5.6.7} = 792$$

$$\eta_5 \div \eta_4 = 24{\cdot}33333333 \ldots$$

$$\eta_6 \div \eta_5 = 466{\cdot}54794520 \ldots$$

$$\eta_7 \div \eta_6 = 204951{\cdot}34925714 \ldots$$

$$\eta_8 \div \eta_7 = 41881671184{\cdot}54776412 \ldots$$

$$\eta_9 \div \eta_8 = 1754062953159389842293{\cdot}346657805 \ldots$$

$$\sqrt{\eta_4} = 4{\cdot}24264068 \ldots$$

$$\sqrt{\eta_5} = 20{\cdot}92844819 \ldots$$

$$\sqrt{\eta_6} = 452{\cdot}04866994 \ldots$$

$$\sqrt{\eta_7} = 204649{\cdot}45227877 \ldots$$

$$\sqrt{\eta_8} = 41881534751{\cdot}051659567667 \ldots$$

Finally, it is interesting to find the asymptotic value of h_x and η_x (the halves of the sharpened Hamiltonian and of the Hypothenusal Numbers), which are ultimately in a ratio of equality to each other, in terms of x. Obviously each of these is ultimately in a ratio of equality with M^{2^x}, where M is a constant to be determined.

Let

$$M = 10^{2^a} \quad \text{and} \quad u_x = 10^{2^{x+a}}.$$

Then, for finite values of x, remembering that (in the preceding notation)

$$p < q^2 \quad \text{and} \quad \lambda > \mu^2,$$

u_x must be intermediate between the corresponding terms of the two series

$$\eta = \tfrac{1}{2},\ 1,\ 3,\ 18,\ 438,\ 204348,\ 41881398318,\ \ldots\ldots,$$

$$h = 2,\ 3,\ 6,\ 24,\ 462,\ 204810,\ 41881603128,\ \ldots\ldots$$

By means of this formula, writing for u_x corresponding values of η and h, and retaining so much of the two corresponding determinations of α as is common to both, we can find α precisely to any desired number of places of decimals, as shown in the following Table, in which 18 and 24 are taken as the terms of place zero in the respective series.

$u_x =$	18,	438,	204348,	41881398318,
$\alpha =$	·32,	·401,	·4088,	·4089863 . . .
$u_x =$	24,	463,	204810,	41881603128,
$\alpha =$	·46,	·413,	·4090,	·4089866 . . .

Hence, if we now change the origin, taking $\frac{1}{2}$ and 2 as the zero terms, we have approximately

$$M^{2^{x+3}} = 10^{2^{x+\alpha}}$$

and

$$8 \log M = 2^{\cdot 408986},$$

which gives

$$M = 1 \cdot 4654433 \ldots ^*$$

As a verification, since $2^3 = 8$, $(1\cdot 46544)^8$ should lie between 18 and 24; and, as a matter of fact, a rough calculation gives

$$(1\cdot 46544)^2 = 2\cdot 1473 \ldots,$$
$$(2\cdot 1473)^2 = 4\cdot 608 \ldots,$$
$$(4\cdot 608)^2 = 21\cdot 234 \ldots,$$

which is about midway between the two limits.—J. J. S.

§ 2.—*Proof of the Formula for the Successive Determination of each in turn of* HAMILTON'S *Numbers from its Antecedents.*

Let

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \ldots = F_0(x),$$
$$2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + \ldots = F_1(x),$$
$$6x^2 + 15x^3 + 29x^4 + 49x^5 + 76x^6 + \ldots = F_2(x),$$
$$36x^3 + 210x^4 + 804x^5 + 2449x^6 + \ldots = F_3(x),$$
$$\ldots \ldots \ldots \ldots \ldots,$$

where the coefficients of the various powers of x are the numbers set out in the triangular Table at the commencement of this paper.

If, in general, we write

$$F_n(x) = a_n x^n + b_n x^{n+1} + c_n x^{n+2} + d_n x^{n+3} + \ldots,$$

the coefficients of $F_{n+1}(x)$, expressed in terms of those of $F_n(x)$, are as follows :—

* See Note 1, p. 312.

$$a_{n+1} = b_n + \frac{a_n(a_n+1)}{1 \,.\, 2}$$

$$b_{n+1} = c_n + a_n b_n + \frac{a_n(a_n+1)(2a_n+1)}{1 \,.\, 2 \,.\, 3}$$

$$c_{n+1} = d_n + a_n c_n + \frac{a_n(a_n+1)}{1 \,.\, 2} b_n + \frac{a_n(a_n+1)(a_n+2)(3a_n+1)}{1 \,.\, 2 \,.\, 3 \,.\, 4}.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Now

$$(1-x)^{-a_n} = 1 + a_n x + \frac{a_n(a_n+1)}{1 \,.\, 2} x^2 + \frac{a_n(a_n+1)(a_n+2)}{1 \,.\, 2 \,.\, 3} x^3 + \ldots,$$

when multiplied by

$$F_n(x) = a_n x^n + b_n x^{n+1} + c_n x^{n+2} + d_n x^{n+3} + \ldots,$$

gives

$$\begin{aligned}(1-x)^{-a_n} F_n(x) = a_n x^n &+ b_n x^{n+1} + c_n x^{n+2} + d_n x^{n+3} + \ldots \\ &+ a_n^2 x^{n+1} + a_n b_n x^{n+2} + a_n c_n x^{n+3} + \ldots \\ &+ \frac{a_n^2(a_n+1)}{1 \,.\, 2} x^{n+2} + \frac{a_n(a_n+1)}{1 \,.\, 2} b_n x^{n+3} + \ldots \\ &+ \frac{a_n^2(a_n+1)(a_n+2)}{1 \,.\, 2 \,.\, 3} x^{n+3} + \ldots \\ &+ \ldots\end{aligned}$$

Comparing this with

$$\begin{aligned}F_{n+1}(x) = b_n x^{n+1} &+ c_n x^{n+2} + d_n x^{n+3} + \ldots \\ &+ \frac{a_n(a_n+1)}{1 \,.\, 2} x^{n+1} + a_n b_n x^{n+2} + a_n c_n x^{n+3} + \ldots \\ &+ \frac{a_n(a_n+1)(2a_n+1)}{1 \,.\, 2 \,.\, 3} x^{n+2} + \frac{a_n(a_n+1)}{1 \,.\, 2} b_n x^{n+3} + \ldots \\ &+ \frac{a_n(a_n+1)(a_n+2)(3a_n+1)}{1 \,.\, 2 \,.\, 3 \,.\, 4} x^{n+3} + \ldots \\ &+ \ldots,\end{aligned}$$

we see that the difference of the two expressions is

$$a_n x^n + \frac{(a_n-1)a_n}{1 \,.\, 2} x^{n+1} + \frac{(a_n-1)\, a_n\, (a_n+1)}{1 \,.\, 2 \,.\, 3} x^{n+2} + \frac{(a_n-1)a_n(a_n+1)(a_n+2)}{1 \,.\, 2 \,.\, 3 \,.\, 4} x^{n+3} + \ldots$$

which is equal to

$$x^{n-1}(1-x)^{-(a_n-1)} - x^{n-1}(1-x).$$

Thus

$$F_{n+1}(x) = (1-x)^{-a_n} \; F_n(x) - x^{n-1}(1-x)^{-a_n+1} + x^{n-1}(1-x).\text{*}$$

Multiplying this equation by $(1-x)^{s_{n+1}}$, where

$$s_{n+1} = a_0 + a_1 + a_2 + \ldots + a_{n-1} + a_n,$$

* See Note 2, p. 312.

we obtain

$$(1-x)^{s_{n+1}} \mathrm{F}_{n+1}(x) = (1-x)^{s_n} \mathrm{F}_n(x) + x^{n-1}(1-x)^{s_{n+1}+1} - x^{n-1}(1-x)^{s_n+1},$$

which gives, when we write successively $n-1$, $n-2$, $n-3$, ... 0 in the place of n,

$$(1-x)^{s_n} \mathrm{F}_n(x) = (1-x)^{s_{n-1}} \mathrm{F}_{n-1}(x) + x^{n-2}(1-x)^{s_n+1} - x^{n-2}(1-x)^{s_{n-1}+1};$$
$$(1-x)^{s_{n-1}} \mathrm{F}_{n-1}(x) = (1-x)^{s_{n-2}} \mathrm{F}_{n-2}(x) + x^{n-3}(1-x)^{s_{n-1}+1} - x^{n-3}(1-x)^{s_{n-2}+1};$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(1-x)^{s_1} \mathrm{F}_1(x) = (1-x)^{s_0} \mathrm{F}_0(x) + x^{-1}(1-x)^{s_1+1} - x^{-1}(1-x)^{s_0+1}.$$

Hence, by addition of these n equations, we find

$$(1-x)^{s_n} \mathrm{F}_n(x) = (1-x)^{s_0} \mathrm{F}_0(x) + x^{n-2}(1-x)^{s_n+1} - x^{-1}(1-x)^{s_0+1}$$
$$+ x^{n-3}(1-x)^{s_{n-1}+2} + x^{n-4}(1-x)^{s_{n-2}+2} + \ldots + x^{-1}(1-x)^{s_1+2},$$

where it has been assumed that it is possible to assign to s_0 (previously undefined) such a value as will make the last of the above n equations, viz.,

$$(1-x)^{s_1} \mathrm{F}_1(x) = (1-x)^{s_0} \mathrm{F}_0(x) + x^{-1}(1-x)^{s_1+1} - x^{-1}(1-x)^{s_0+1},$$

identically true. That this can be done is obvious; for, if in that equation we write for $\mathrm{F}_1(x)$, $\mathrm{F}_0(x)$, and s_1 their values, viz.,

$$\mathrm{F}_1(x) = (1-x)^{-2} - 1, \qquad \mathrm{F}_0(x) = (1-x)^{-1}, \qquad \text{and } s_1 = a_0 = 1,$$

then, on making $s_0 = 0$, the equation becomes

$$(1-x)^{-1} - (1-x) = (1-x)^{-1} + x^{-1}(1-x)\,(1-x-1).$$

Thus the general value of $\mathrm{F}_n(x)$ is given by the equation

$$(1-x)^{s_n} \mathrm{F}_n(x) = (1-x)^{-1} + x^{n-2}(1-x)^{s_n+1} - x^{-1}(1-x)$$
$$+ x^{n-3}(1-x)^{s_{n-1}+2} + x^{n-4}(1-x)^{s_{n-2}+2} + \ldots + x^{-1}(1-x)^{s_1+2},$$

which is equivalent to

$$(1-x)^{s_n} \mathrm{F}_n(x) - (1-x)^{-1} + x^{-1}(1-x) - x^{n-1}(1-x)^{s_n+1}$$
$$= x^{n-2}(1-x)^{s_n+2} + x^{n-3}(1-x)^{s_{n-1}+2} + x^{n-4}(1-x)^{s_{n-2}+2} + \ldots + x^{-1}(1-x)^{s_1+2},$$

where, a_0, a_1, a_2, a_3, ... being the Hypothenusal Numbers 1, 2, 6, 36, ... we have

$$s_1 = a_0 = 1,$$
$$s_2 = a_0 + a_1 = 3,$$
$$s_3 = a_0 + a_1 + a_2 = 9,$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . ;$$

i.e., the successive values of $s_n + 2$ are the Hamiltonian Numbers 3, 5, 11, 47 . . .

Now $F_n(x) = a_n x^n + \ldots$, so that the coefficient of x^n in $(1-x)^{s_n} F_n(x)$ is the same as the coefficient of x^n in $F_n(x)$, viz., a_n. Consequently, equating coefficients of x^n on each side of the equation just obtained, we find

$$a_n - 1 + (s_n + 1) = \frac{(s_n + 2)(s_n + 1)}{1.2} - \frac{(s_{n-1} + 2)(s_{n-1} + 1)s_{n-1}}{1.2.3}$$
$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$+ (-)^{n+1} \frac{(s_1 + 2)(s_1 + 1) \ldots (s_1 + 2 - n)}{1.2.3 \ldots (n+1)}$$

Remembering that

$$a_n + s_n = s_{n+1},$$

if we call the Hamiltonian Number $s_n + 2$, H_n, the above relation may be written thus:

$$H_{n+1} - 2 = \frac{H_n(H_n - 1)}{1.2} - \frac{H_{n-1}(H_{n-1} - 1)(H_{n-1} - 2)}{1.2.3}$$
$$+ \frac{H_{n-2}(H_{n-2} - 1)(H_{n-2} - 2)(H_{n-2} - 3)}{1.2.3.4}$$
$$- \quad .$$
$$+ (-)^{n+1} \frac{H_1(H_1 - 1)(H_1 - 2) \ldots (H_1 - n)}{1.2.3 \ldots (n+1)}.$$

To obtain Professor SYLVESTER's modification of this formula given in the preceding portion of this memoir, we multiply the equation from which it was obtained by $1 - x$ before proceeding to equate coefficients. Thus we have to equate coefficients of x^n on both sides of

$$(1-x)^{s_n+1} F_n(x) - 1 + x^{-1}(1-x)^2 - x^{n-1}(1-x)^{s_n+2}$$
$$= x^{n-2}(1-x)^{s_n+3} + x^{n-3}(1-x)^{s_{n-1}+3} + x^{n-4}(1-x)^{s_{n-2}+3} + \ldots + x^{-1}(1-x)^{s_1+3}.$$

Or, writing

$$s_n + 3 = E_n,$$

we equate coefficients on both sides of

$$(1-x)^{E_n-2} F_n(x) - 1 + x^{-1}(1-x)^2 - x^{n-1}(1-x)^{E_n-1}$$
$$= x^{n-2}(1-x)^{E_n} + x^{n-3}(1-x)^{E_{n-1}} + x^{n-4}(1-x)^{E_{n-2}} + \ldots x^{-1}(1-x)^{E_1}.$$

This equation is easily transformed into

$$(1-x)^{E_0} + x(1-x)^{E_1} + x^2(1-x)^{E_2} + \ldots + x^n(1-x)^{E_n}$$
$$= 1 - 2x + x^2(1-x)^{E_n - 2} F_n(x) - x^{n+1}(1-x)^{E_n - 1},$$

from which, as Professor SYLVESTER has pointed out in this memoir, by equating coefficients of all powers of x from 0 to n, we can obtain the successive values of E_n.

The general formula

$$1 - E_{n-1} + \frac{E_{n-2}(E_{n-2}-1)}{1.2} - \ldots + (-)^n \frac{E_0(E_0-1)\ldots(E_0-n+1)}{1.2\ldots n} = 0$$

arises from equating the coefficients of x^n.—J. H.*

§ 3.† *Sequel to the Asymptotic Theory contained in* § 1.

The relation

$$p = q^2 - \tfrac{2}{3}q^{\frac{3}{2}}, \text{ etc.}$$

previously obtained supplies only the two first terms of the remarkable asymptotic development

$$\frac{q^2 - p}{q} = \tfrac{2}{3}\left(q^{\frac{1}{2}} + q^{\frac{1}{4}} + q^{\frac{1}{8}} + \ldots + q^{(\frac{1}{2})^i}\right) + \Xi,$$

where i is any assigned integer and Ξ is of a lower order of magnitude than the lowest power of q in the series which precedes it. This may be easily established as follows:—

By the scale of relation proved in the preceding section we have

$$p = q^2 - \tfrac{2}{3}r^3 + \frac{s^4}{3} + \ldots$$
$$= q^2 - \tfrac{2}{3}r^3 + \text{terms whose maximum order is that of } r^2.$$

Let, now,

$$p = q^2 - \tfrac{2}{3}q^{\frac{3}{2}} - \tfrac{2}{3}hq^{\alpha} - \tfrac{2}{3}kq^{\beta} - \tfrac{2}{3}lq^{\gamma} \ldots;$$

therefore

$$q = r^2 - \tfrac{2}{3}r^{\frac{3}{2}} - \tfrac{2}{3}hr^{\alpha} - \tfrac{2}{3}kr^{\beta} - \tfrac{2}{3}lr^{\gamma} \ldots$$

and

$$p = q^2 - \tfrac{2}{3}r^3\left(1 - r^{-\frac{1}{2}} - hr^{\alpha-2} - kr^{\beta-2} - lr^{\gamma-2} \ldots\right) + \ldots$$
$$- \tfrac{2}{3}hr^{2\alpha} - \tfrac{2}{3}kr^{2\beta} - \tfrac{2}{3}lr^{2\gamma} \ldots$$
$$= q^2 - \tfrac{2}{3}r^3 + \tfrac{2}{3}\left(r^{\frac{5}{2}} + hr^{\alpha+1} + kr^{\beta+1} + lr^{\gamma+1} + \ldots\right) + \ldots$$
$$- \tfrac{2}{3}hr^{2\alpha} - \tfrac{2}{3}kr^{2\beta} - \tfrac{2}{3}lr^{2\gamma} - \quad \ldots\ldots$$

Therefore

$$h = 1, \quad k = 1, \quad l = 1, \quad m = 1, \quad \ldots$$
$$2\alpha = \tfrac{5}{2}, \quad 2\beta = 1 + \alpha, \quad 2\gamma = 1 + \beta, \quad 2\delta = 1 + \gamma, \quad \ldots$$

i.e.,

$$\alpha = \tfrac{5}{4}, \quad \beta = \tfrac{9}{8}, \quad \gamma = \tfrac{17}{16}, \quad \delta = \tfrac{33}{32}, \quad \ldots$$

* See Note 3, p. 312.

† Received July 28, 1887.

and thus

$$p = q^2 - \tfrac{2}{3}q\,(q^{\frac{1}{2}} + q^{\frac{1}{4}} + q^{\frac{1}{8}} + q^{\frac{1}{16}} + \ldots + q^{(\frac{1}{2})^i}) + \Xi,$$

as was to be shown.*

* This theorem may be rigorously demonstrated, and reduced to a more precise analytical form, as follows:—

For the sake of brevity, we may call $-p/q + q$ the relative deficiency of p, and denote it by Δ.

First it may be noticed that, if in the equation

$$\mathrm{F}(q) = \Sigma_0^\infty (q^{2^{-s}} - q^{-2^{-s}})$$

we write $\log q = k$,

$$\mathrm{F}(q) = 2\left(k + \frac{k^3}{1\,.\,2\,.\,3\,.\,7} + \frac{k^5}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,31} + \frac{k^7}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,127} + \ldots\right),$$

which is always convergent.

Moreover, the value of $\mathrm{F}(q)$ may be calculated for any given value of q within close limits. For, if we call U the right-hand branch of the series in q, beginning with $z - z^{-1}$, the terms of U will easily be seen to lie between those of two geometrical series of which $z - z^{-1}$ is the first term, and of one of which $\frac{1}{2}$, and of the other $(z^{\frac{1}{2}} + z^{-\frac{1}{2}})^{-1}$, is the common ratio.

Hence U is intermediate between $2\,(z^2 - 1)/z$ and $(z^2 - 1)\;(z + 1)/z\,(z - z^{\frac{1}{2}} + 1)$.

[The difference between these limits, it may be parenthetically observed, is

$$(z - z^{-1})\,\frac{(z^{\frac{1}{4}} - z^{-\frac{1}{4}})^2}{z^{\frac{1}{2}} - 1 + z^{-\frac{1}{2}}},$$

which, when z is nearly unity (the limit to which $q^{(\frac{1}{2})^s}$ converges), is nearly equal to $\frac{1}{16}(z - z^{-1})^3$; *i.e.*, if $z = 1 + \tau$, the difference between the limits (for τ small) is very near to $\tau^3/2$.]

Now on p. 309 (*post*) it is shown that $\sqrt{q} - r + s - \frac{2}{3}\sqrt{r} = \epsilon$, and that, when the rank of q is taken indefinitely great, ϵ converges to $\frac{1}{4}$. Hence ϵ always lies between finite limits.

[For, in general, x being any one of a series of increasing numbers, and $\psi(x)$ a function of x which is always finite for finite values of x, but ultimately converges to c, by taking for x a value of L sufficiently great, we make the series of terms for $x > \mathrm{L}$ intermediate between $c + \delta$ and $c - \delta$, where δ is any assigned positive quantity; and consequently, if μ, ν, are the greatest and least values of $\psi(x)$ when x does not exceed L, the greater of the two values, $c + \delta$, μ, and the lesser of the two, $c - \delta$, ν, will be superior and inferior limits to the value of $\psi(x)$ for all values of x.]

Hence, writing

$$\begin{aligned}
&\sqrt{p} - q + r - \tfrac{2}{3}\sqrt{q} = \epsilon_1,\\
&\sqrt{q} - r + s - \tfrac{2}{3}\sqrt{r} = \epsilon_2,\\
&\sqrt{r} - s + t - \tfrac{2}{3}\sqrt{s} = \epsilon_3,\\
&\ldots\ldots\ldots\ldots,\\
&\sqrt{6} - 3 + 2 - \tfrac{2}{3}\sqrt{3} = \epsilon_{x-1},
\end{aligned}$$

we obtain, by summation,

$$\sqrt{p} - q + \tfrac{1}{3}(\sqrt{q} + \sqrt{r} + \sqrt{s} + \ldots + \sqrt{6}) = \Sigma\epsilon - 2 + \tfrac{2}{3}\sqrt{3},$$

and, consequently,

$$\sqrt{p} - q + \tfrac{1}{3}(\sqrt{q} + \sqrt{r} + \sqrt{s} + \ldots + \sqrt{6}) = \rho x,$$

where ρ is always a finite quantity lying between determinable limits. But again (p. 307)—

$$p = (q - \theta\sqrt{q})^2,$$

where θ (whose ultimate value is $\frac{1}{3}$) is always a proper fraction. Hence

$$\frac{q^2 - p}{q} = 2\,(q - \sqrt{p}) - \theta^2.$$

It is interesting to notice that the formula *apparently* remains arithmetically true for *finite* values of p and q, provided that q is not less than 24, when we replace each

Hence, from what has been shown above,

$$\frac{q^2-p}{q} = \tfrac{2}{3}(\sqrt{q} + \sqrt{r} + \sqrt{s} + \ldots + \sqrt{6}) - 2\rho' x.$$

In this equation we may write

$$\left.\begin{array}{l}\sqrt{r} = q^{\frac{1}{4}} + k_1 \\ \sqrt{s} = q^{\frac{1}{8}} + k_2 \\ \sqrt{t} = q^{\frac{1}{16}} + k_3 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\end{array}\right\} (x-3 \text{ equations}),$$

where $k_1, k_2, k_3, \ldots$ are all of them finite (and, as a matter of fact, of no consequence for our immediate object, positive proper fractions). For, ultimately,

$$k_1 = \sqrt{r} - q^{\frac{1}{4}} = \frac{r - q^{\frac{1}{2}}}{r^{\frac{1}{2}} + q^{\frac{1}{4}}} = \frac{\theta_1 r^{\frac{1}{2}}}{r^{\frac{1}{2}} + q^{\frac{1}{4}}} = \tfrac{1}{2}\theta_1 = \tfrac{1}{6} \text{ (see p. 307)},$$

and consequently the finiteness of each k is a direct inference from the general principle previously applied in the case of the ϵ's.

Applying this result to the equation previously given, it follows that $q^{\frac{1}{2}} + q^{\frac{1}{4}} + \ldots + q^{(\frac{1}{2})^{x-2}} = \frac{3}{2}\Delta - vx$ (where v is finite) $= \mathrm{F}(q) + (q^{-\frac{1}{2}} + q^{-\frac{1}{4}} + \ldots + q^{-(\frac{1}{2})^{x-2}}) - \{(z - z^{-1}) + (z^{\frac{1}{2}} - z^{-\frac{1}{2}}) + (z^{\frac{1}{4}} - z^{-\frac{1}{4}}) + \ldots\}$, where z lies between 1 and 2.

The series of negative powers of q is obviously less than x, and the z-series, which follows it, is less than the finite quantity $2(z - 1/z)$, *i.e.*, $< 2(2 - \frac{1}{2})$. Hence $\frac{3}{2}\Delta = \mathrm{F}(q) + \Theta x$, where Θ is a number lying between fixed limits, and x, the rank of q, is of the same order of magnitude as log log q. This equation contains as a consequence the asymptotic theorem to be proved; for, using i to denote any positive integer,

$$\tfrac{3}{2}\Delta - \Sigma_1^i\, q^{(\frac{1}{2})^i} = \mathrm{F}(q) - \Sigma_1^i\, q^{(\frac{1}{2})^i} - \Theta x = q^{(\frac{1}{2})^{i+1}} + \sum_{s=i+2}^{s=\infty} (q^{(\frac{1}{2})^s} - q^{-(\frac{1}{2})^s}) - \Sigma_0^i\, 1/q^{(\frac{1}{2})^{i+1}} - \Theta x.$$

Hence, remembering that x is of the same order of magnitude as log log q, and that

$$\sum_{s=i+2}^{s=\infty} (q^{(\frac{1}{2})^s} - q^{-(\frac{1}{2})^s}) < 2(q^{(\frac{1}{2})^{i+2}} - q^{-(\frac{1}{2})^{i+2}}),$$

which is of a lower order of magnitude than $q^{(\frac{1}{2})^{i+1}}$, it follows that $\frac{3}{2}\Delta - \Sigma_1^i\, q^{(\frac{1}{2})^i}$ for all values of i is ultimately in a ratio of equality with $q^{(\frac{1}{2})^{i+1}}$, which is the theorem to be proved.

We have thought it desirable to obtain the formula $\frac{3}{2}\Delta = \mathrm{F}q + \Theta x$ for its own sake, but, so far as regards the proof in question, that might be obtained more expeditiously from the expression given for $3\Delta/2 - vx$ without introducing the series $\mathrm{F}q$.

It is easy to ascertain the ultimate value to which Θ converges. In the first place, the series of fractions $1/q^{\frac{1}{2}} + 1/q^{\frac{1}{4}} + 1/q^{\frac{1}{8}} + \ldots$ to $x - 2$ terms (where x is the rank of q) may be shown to be always finite, and consequently, when divided by x, converges to zero.

For we know that $(p - q) > (q - r)^2 > (r - s)^4 \ldots > (6 - 3)^{2^{x-2}}$. Hence the last term of the series $q^{\frac{1}{2}}, q^{\frac{1}{4}}, q^{\frac{1}{8}} \ldots$ (viz., $q^{(\frac{1}{2})^{x-2}}$) > 3. Hence the finite series $1/q^{\frac{1}{2}} + 1/q^{\frac{1}{4}} + 1/q^{\frac{1}{8}} + \ldots$ for a double *à fortiori* reason is less than the infinite geometrical series $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots < \frac{1}{2}$.

[In fact, from § 1 (p. 299) it may easily be shown that the last term of the series $q^{\frac{1}{2}}, q^{\frac{1}{4}}, q^{\frac{1}{8}} \ldots > \mathrm{M}^4 > (1{\cdot}465)^4 > 4{\cdot}608$, so that the sum is really less than $\frac{1}{3{\cdot}608}$.]

Hence, retracing the steps by which Θ has been obtained, and observing that ρ' differs from ρ by a finite multiple of $1/x$, we have ultimately $\Theta = v = k - 3\rho' = k - 3\rho = k - 3\epsilon = \frac{1}{6} - \frac{3}{4} = -\frac{7}{12}$. If, then (using u_x to denote the half of the sharpened x^{th} Hamiltonian number), we write $u_x - 1/u_x = v_x$, and understand by $\mathrm{G}(t - 1/t)$ the infinite series $(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) + (t^{\frac{1}{4}} - t^{-\frac{1}{4}}) + (t^{\frac{1}{8}} - t^{-\frac{1}{8}}) + \ldots$, it is easily seen that the *principal* part of $\sqrt{(v_{x+1})}$, regarded as a function of v_x and x, is $v_x - \frac{1}{3}\mathrm{G}v_x + \frac{7}{36}x$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

term in the formula by its integer portion, and in the series on the right stop at the term immediately preceding the first term for which

$$\mathrm{E}q^{(\frac{1}{2})} = 1.$$

Thus, when

$$p = 462 \quad \text{and} \quad q = 24,$$

we have

$$\mathrm{E}\left(\frac{q^2 - p}{q}\right) = \mathrm{E}\left(\frac{576 - 462}{24}\right) = \mathrm{E}\left(\frac{114}{24}\right) = 4,$$

and

$$\mathrm{E}\,\{\tfrac{2}{3}\,(\mathrm{E}q^{\frac{1}{2}} + \mathrm{E}q^{\frac{1}{4}})\,\} = \mathrm{E}\,\{\tfrac{2}{3}\,(4 + 2)\,\} = 4.$$

So also, when

$$p = 41881603128, \qquad q = 204810,$$

$$\mathrm{E}\left(\frac{q^2 - p}{q}\right) = 319,$$

and

$$\mathrm{E}\,\{\tfrac{2}{3}\,(\mathrm{E}q^{\frac{1}{2}} + \mathrm{E}q^{\frac{1}{4}} + \mathrm{E}q^{\frac{1}{8}} + \mathrm{E}q^{\frac{1}{16}})\,\} = \mathrm{E}\,\{\tfrac{2}{3}\,(452 + 21 + 4 + 2)\,\} = \mathrm{E}\,(\tfrac{958}{3}) = 319.$$

But, if we had included the term $\mathrm{E}q^{\frac{1}{32}}$, the result would have been

$$\mathrm{E}\,\{\tfrac{2}{3}\,(452 + 21 + 4 + 2 + 1)\,\} = 320.$$

Again, when

$$p = 3076736843548289379224261404637538760216584,$$

$$q = 1754062953145429399086,$$

$$\mathrm{E}\left(\frac{q^2 - p}{q}\right) = 27921159919,$$

and

$$\mathrm{E}\,\{\tfrac{2}{3}\,(\mathrm{E}q^{\frac{1}{2}} + \mathrm{E}q^{\frac{1}{4}} + \mathrm{E}q^{\frac{1}{8}} + \mathrm{E}q^{\frac{1}{16}} + \mathrm{E}q^{\frac{1}{32}} + \mathrm{E}q^{\frac{1}{64}})\,\}$$

$$= \mathrm{E}\,\{\tfrac{2}{3}\,(41881534751 + 204649 + 452 + 21 + 4 + 2)\,\} = 27921159919.*$$

We will now proceed to consider afresh the asymptotic development of any *Hypothenusal* Number $p - q$ in terms of its antecedent $q - r$, and to reduce to apodictic certainty results which in the first section were partly obtained by observation. It has already been shown in that section that

$$p > q^2 - \tfrac{128}{81}\,q^{\frac{3}{2}} - \frac{q}{2}$$

when p is not lower than 204810 in the scale 2, 3, 6, 24, 462, 204810, . . . , *i.e.*, when q is not less than 462.

Hence

$$p > q^2 - 2q^{\frac{3}{2}} + q + (\tfrac{34}{81}\,q^{\frac{3}{2}} - \tfrac{3}{2}\,q),$$

* The authors must be understood merely to affirm the *possibility* of the theorem being true, and to offer no opinion on the strength of the presumption raised that it is so.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

or, since $\frac{34}{81} q^{\frac{3}{2}} - \frac{3}{2} q$ is a positive quantity,

$$p > (q - \sqrt{q})^2,$$

at all events when $q =$ or > 462.

It will be found also on trial that this formula remains true for all the values of q inferior to 462.

Thus

$$462 > (24 - \sqrt{24})^2,$$
$$24 > (\ 6 - \sqrt{6})^2,$$
$$6 > (\ 3 - \sqrt{3})^2,$$
$$3 > (\ 2 - \sqrt{2})^2.$$

Hence, universally,

$$p > (q - \sqrt{q})^2.*$$

But we know that

$$p < q^2.$$

We may therefore write

$$p = (q - \theta \sqrt{q})^2,$$

where θ is some quantity between 0 and 1.

Similarly,

$$q = (r - \theta_1 \sqrt{r})^2,$$
$$r = (s - \theta_2 \sqrt{s})^2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot,$$

where θ_1, θ_2, . . . are also positive fractions.

When p and q become infinite,

$$\frac{q^2 - p}{q^{\frac{3}{2}}} = \tfrac{2}{3} = 2\theta.$$

Hence the ultimate value of θ is $\frac{1}{3}$. Similarly, θ_1, θ_2, . . . all of them converge to the value $\frac{1}{3}$.

This agrees with the result previously demonstrated (p. 295), and is the starting point of all that follows.

We know that letters p, q, r, s, . . ., being used to denote the halves of the augmented Hamiltonian Numbers, they are connected by the scale of relation

$$p = \tfrac{1}{2} + \frac{q(2q-1)}{2} - \frac{r(2r-1)(2r-2)}{2 \cdot 3} + \mathrm{S} - \mathrm{T},$$

* Had this inequality been true only for values of q sufficiently great, it would have been enough for the purposes of the text.

where

$$S = \frac{s(2s-1)(2s-2)(2s-3)}{2.3.4}$$

and T stands for the remaining terms, involving

$$t, \; u, \; v, \; \ldots$$

Considering

$$q, \; r, \; s, \; t, \; \ldots$$

to be of the order

$$1, \; \tfrac{1}{2}, \; \tfrac{1}{4}, \; \tfrac{1}{8}, \; \ldots,$$

we may reject the term $\frac{1}{2}$, which is of zero order, and write

$$p = q^2 - \tfrac{2}{3}r^3; \qquad -\frac{q}{2} + r^2 - \frac{r}{3} + S - T.$$

Hence, rejecting terms of order less than $\frac{3}{2}$ (which have, however, to be retained in obtaining the subsequent approximations),

$$\left.\begin{array}{r} (p-q) \\ -(q-r)^2 \end{array}\right\} = \left\{\begin{array}{ll} q^2 - \tfrac{2}{3}r^3; & \quad -\tfrac{3}{2}q + r^2 - \dfrac{r}{3} + S - T \\ -q^2 + 2qr; & \quad -r^2 \end{array}\right.$$

$$= (2qr - \tfrac{2}{3}r^3);$$

i.e.,

$$(p-q) - (q-r)^2 = \tfrac{4}{3}q^{\frac{3}{2}}$$

when q is infinite.

Again, writing for S its expanded value, viz.,

$$\frac{s^4}{3} - s^3 + \tfrac{11}{12}s^2 - \frac{s}{4},$$

we have

$$\left.\begin{array}{rr} & (p-q) \\ - & (q-r)^2 \\ -\tfrac{4}{3} & (q-r)^{\frac{3}{2}} \end{array}\right\} = \left\{\begin{array}{ll} 2qr - \tfrac{2}{3}r^3 - \tfrac{4}{3}q^{\frac{3}{2}} & \text{Order } \tfrac{3}{2}; \\ +2q^{\frac{1}{2}}r - \tfrac{3}{2}q + \tfrac{1}{3}s^4 & \quad ,, \quad 1; \\ -\tfrac{1}{2}q^{-\frac{1}{2}}r^2 - \dfrac{r}{3} - s^3 + \tfrac{11}{12}s^2 - \tfrac{1}{4}s - T & \quad ,, \quad < 1, \end{array}\right.$$

rejecting the terms $q^{-\frac{3}{2}}r^3$, $q^{-\frac{5}{2}}r^4$, ... in the expansion of $(q-r)^{\frac{3}{2}}$ because the order of none of them is superior to zero.

We now write

$$q = (r - \theta_1\sqrt{r})^2,$$

so that

$$2qr - \tfrac{2}{3}r^3 - \tfrac{4}{3}q^{\frac{3}{2}} = (2r^3 - 4\theta_1 r^{\frac{5}{2}} + 2\theta_1{}^2 r^2) - \tfrac{2}{3}r^3 - (\tfrac{4}{3}r^3 - 4\theta_1 r^{\frac{5}{2}} + 4\theta_1{}^2 r^2 - \tfrac{4}{3}\theta_1{}^3 r^{\frac{3}{2}})$$

$$= -2\theta_1{}^2 r^2 + \tfrac{4}{3}\theta_1{}^3 r^{\frac{3}{2}}.$$

Hence

$$\left.\begin{array}{r} (p-q) \\ -(q-r)^2 \\ -\tfrac{4}{3}(q-r)^{\frac{3}{2}} \end{array}\right\} = \left\{\begin{array}{ll} -2\theta_1{}^2 r^2 + 2q^{\frac{1}{2}}r - \tfrac{3}{2}q + \tfrac{1}{3}s^4 & \text{Order } 1; \\ +\tfrac{4}{3}\theta_1{}^3 r^{\frac{3}{2}} - s^3 & \quad ,, \quad \tfrac{3}{4}; \\ -\tfrac{1}{2}q^{-\frac{1}{2}}r^2 - \dfrac{r}{3} + \tfrac{11}{12}s^2 - \tfrac{1}{4}s - T & \quad ,, \quad < \tfrac{3}{4}. \end{array}\right.$$

Since

$$q = r^2 = s^4 \text{ (ultimately)},$$

the terms of Order 1 (which are the only ones with which we have to do at present) are ultimately equal to

$$(-2\theta_1^2 + 2 - \tfrac{3}{2} + \tfrac{1}{3})q\,;$$

or, giving θ_1 its ultimate value $\frac{1}{3}$, to $\frac{11}{18}q$, or to the same order of approximation to $\frac{11}{18}(q-r)$.

Hence, ultimately,

$$(p-q) = (q-r)^2 + \tfrac{4}{3}(q-r)^{\frac{3}{2}} + \tfrac{11}{18}(q-r).^*$$

We use this result to obtain a closer approximation to $\sqrt{q}$ than $r - \theta_1\sqrt{r}$, and to find the relation between the general values of θ_1 and θ_2.

Thus, assuming

$$\sqrt{q-r} = r - s + \tfrac{2}{3}\sqrt{r-s} + k,$$

we have, ultimately,

$$q - r = (r-s)^2 + \tfrac{4}{3}(r-s)^{\frac{3}{2}} + (\tfrac{4}{9} + 2k)(r-s)$$
$$= (r-s)^2 + \tfrac{4}{3}(r-s)^{\frac{3}{2}} + \tfrac{11}{18}(r-s).$$

Consequently, as r becomes indefinitely great, k converges to the value $\frac{1}{2}(\frac{11}{18} - \frac{4}{9}) = \frac{1}{12}$.

Now

$$\sqrt{q-r} = \sqrt{q} - \tfrac{1}{2}\frac{r}{\sqrt{q}} \ldots = \sqrt{q} - \tfrac{1}{2} \text{ ultimately};$$

and similarly

$$\sqrt{r-s} = \sqrt{r} - \tfrac{1}{2} \text{ ultimately}.$$

Hence, ultimately,

$$\sqrt{q} = r - s + \tfrac{2}{3}\sqrt{r} + \tfrac{1}{12} + \tfrac{1}{2} - \tfrac{1}{3} = r - s + \tfrac{2}{3}\sqrt{r} + \tfrac{1}{4}.$$

We may therefore write

$$\sqrt{q} = r - s + \tfrac{2}{3}\sqrt{r} + \epsilon \text{ (where ultimately } \epsilon = \tfrac{1}{4}).$$

But

$$\sqrt{q} = r - \theta_1\sqrt{r},$$

and therefore

$$\theta_1\sqrt{r} = s - \tfrac{2}{3}\sqrt{r} - \epsilon.$$

Moreover

$$\sqrt{r} = s - \theta_2\sqrt{s},$$

whence it follows that

$$\theta_1\sqrt{r} = \tfrac{1}{3}s + \tfrac{2}{3}\theta_2\sqrt{s} - \epsilon \text{ (where } \epsilon = \tfrac{1}{4} \text{ ultimately)}.$$

* As previously obtained by observation in § 1 (pp. 296, 297). It will, of course, be understood that in the above and similar passages the sign = is to be interpreted to mean "is in a ratio of equality with."

Resuming the development of $(p-q)$ in terms of $(q-r)$, we have

$$\left.\begin{array}{r} (p-q) \\ -\quad (q-r)^2 \\ -\frac{4}{3}\,(q-r)^{\frac{3}{2}} \\ -\frac{11}{18}\,(q-r) \end{array}\right\} = \left\{\begin{array}{lll} -2\theta_1 r^2 + 2q^{\frac{1}{2}}r - \frac{19}{9}q + \frac{s^4}{3} & \text{Order} & 1. \\ +\frac{4}{3}\theta_1{}^3 r^{\frac{3}{2}} - s^3 & \quad ,, & \frac{3}{4}. \\ & & \\ -\frac{1}{2}q^{-\frac{1}{2}}r^2 + \frac{5}{18}r + \frac{11}{12}s^2 - \frac{s}{4} - \mathrm{T} & \quad ,, & < \frac{3}{4}. \end{array}\right.$$

The terms of order inferior to $\frac{3}{4}$ are of no value for present purposes, and are only retained for the benefit of those who may wish to carry on the work.

To reduce the terms of Order 1, we write, in succession,

$$q = (r - \theta_1\sqrt{r})^2,$$
$$\theta_1\sqrt{r} = \tfrac{1}{3}s + \tfrac{2}{3}\theta_2\sqrt{s} - \epsilon,$$
$$r = (s - \theta_2\sqrt{s})^2.$$

Thus

$$\frac{s^4}{3} - 2\theta_1{}^2 r^2 + 2q^{\frac{1}{2}}r - \tfrac{19}{9}q$$
$$= \frac{s^4}{3} - 2\theta_1{}^2 r^2 + 2r^2 - \tfrac{19}{9}r^2\,;\; - 2\theta_1 r^{\frac{3}{2}} + \tfrac{38}{9}\theta_1 r^{\frac{3}{2}}\,;\; - \tfrac{19}{9}\theta_1{}^2 r$$
$$= \frac{s^4}{3} - \frac{r^2}{9} - 2\theta_1{}^2 r^2\,;\; + \tfrac{20}{9}\theta_1 r^{\frac{3}{2}}\,;\; - \tfrac{19}{9}\theta_1{}^2 r$$
$$= \frac{s^4}{3} - \frac{r^2}{9} - 2r\left(\frac{s}{3} + \tfrac{2}{3}\theta_2\sqrt{s}\right)^2;\; + 4\epsilon r\left(\frac{s}{3} + \tfrac{2}{3}\theta_2\sqrt{s}\right) + \tfrac{20}{9}\theta_1 r^{\frac{3}{2}}\,;\; - 2\epsilon^2 r - \tfrac{19}{9}\theta_1{}^2 r$$
$$= \frac{s^4}{3} - \tfrac{1}{9}(s^4 - 4\theta_2 s^{\frac{7}{2}} + 6\theta_2{}^2 s^3 - 4\theta_2{}^3 s^{\frac{5}{2}} + \theta_2{}^4 s^2) - \tfrac{2}{9}s^2(s^2 + 4\theta_2 s^{\frac{3}{2}} + 4\theta_2{}^2 s)$$
$$+ \tfrac{4}{9}\theta_2 s^{\frac{3}{2}}(s^2 + 4\theta_2 s^{\frac{3}{2}} + 4\theta_2{}^2 s) - \tfrac{2}{9}\theta_2{}^2 s(s^2 + 4\theta_2 s^{\frac{3}{2}} + 4\theta_2{}^2 s)\,;$$
$$+ \tfrac{4}{3}\epsilon rs + \tfrac{20}{9}\theta_1 r^{\frac{3}{2}}\,;\; + \tfrac{8}{3}\epsilon\theta_2 r\sqrt{s} - 2\epsilon^2 r - \tfrac{19}{9}\theta_1{}^2 r$$
$$= \tfrac{4}{3}\epsilon rs + \tfrac{20}{9}\theta_1 r^{\frac{3}{2}} \qquad\qquad \text{Order} \quad \tfrac{3}{4}$$
$$+ \tfrac{4}{3}\theta_2{}^3 s^{\frac{5}{2}} + \tfrac{8}{3}\epsilon\theta_2 r\sqrt{s} - \theta_2{}^4 s^2 - 2\epsilon^2 r - \tfrac{19}{9}\theta_1{}^2 r \qquad ,, \quad < \tfrac{3}{4}.$$

Hence

$$\left.\begin{array}{r} (p-q) \\ -\quad (q-r)^2 \\ -\frac{4}{3}\,(q-r)^{\frac{3}{2}} \\ -\frac{11}{18}\,(q-r) \end{array}\right\} = \left\{\begin{array}{lll} \frac{4}{3}\theta_1{}^3 r^{\frac{3}{2}} - s^3 + \frac{4}{3}\epsilon rs + \frac{20}{9}\theta_1 r^{\frac{3}{2}} & \text{Order} & \frac{3}{4} \\ & & \\ -\mathrm{T} + \frac{4}{3}\theta_2{}^3 s^{\frac{5}{2}} + \frac{8}{3}\epsilon\theta_2 r s^{\frac{1}{2}} & \quad ,, & \frac{5}{8} \\ -\frac{1}{2}q^{-\frac{1}{2}}r^2 + \frac{5}{18}r + \frac{11}{12}s^2 - \frac{s}{4} - \theta_2{}^4 s^2 - 2\epsilon^2 r - \frac{19}{9}\theta_1{}^2 r & \quad ,, & < \frac{5}{8}. \end{array}\right.$$

Here the terms of Order $\frac{3}{4}$ are ultimately equal to

$$\left(\tfrac{4}{3}\,\theta_1^{\,3} - 1 + \tfrac{4}{3}\,\epsilon + \tfrac{20}{9}\,\theta_1\right) q^{\frac{3}{4}},$$

which, when θ_1 and ϵ receive their ultimate values, $\frac{1}{3}$ and $\frac{1}{4}$, becomes

$$\left(\tfrac{4}{81} - 1 + \tfrac{1}{3} + \tfrac{20}{27}\right) q^{\frac{3}{4}} = \tfrac{10}{81}\, q^{\frac{3}{4}}.*$$

From this it follows immediately that (rejecting terms of an order of magnitude inferior to that $(q-r)^{\frac{3}{4}}$)

$$p - q = (q-r)^2 + \tfrac{4}{3}\,(q-r)^{\frac{3}{2}} + \tfrac{11}{18}\,(q-r) + \tfrac{10}{81}\,(q-r)^{\frac{3}{4}}.$$

The law of the indices in the complete development is easily deduced from the relation

$$p = \tfrac{1}{2} + \frac{q\,(2q-1)}{2} - \frac{r\,(2r-1)\,(2r-2)}{2\,.\,3} + \frac{s\,(2s-1)\,(2s-2)\,(2s-3)}{2\,.\,3\,.\,4} - \ldots$$

The terms carrying the arguments

$$q^2,\; q,\; r^3,\; r^2,\; r,\; s^4,\; s^3,\; s^2,\; s,\; t^5, \ldots$$

furnish the indices

$$2,\; 1,\; \tfrac{3}{2},\; 1,\; \tfrac{1}{2},\; 1,\; \tfrac{3}{4},\; \tfrac{1}{2},\; \tfrac{1}{4},\; \tfrac{5}{8}, \ldots,$$

which, arranged in order of magnitude, become

$$2,\; \tfrac{3}{2},\; 1,\; \tfrac{3}{4},\; \tfrac{5}{8},\; \tfrac{1}{2},\; \tfrac{3}{8},\; \tfrac{5}{16},\; \tfrac{1}{4}, \ldots$$

Thus, calling $p-q$ and $q-r$ y and x respectively, the expansion for y in terms of x will be of the form

$$y = \Sigma \mathrm{A} x^{\frac{2m+1}{2^n}},$$

where n has all values from 0 to ∞, and $2m+1$ does not exceed $n+2$, *i.e.*, m has all positive values from 0 to $n/2$ or $\frac{1}{2}(n+1)$, according as n is even or odd.

But, besides this expressed portion of the development of a Hypothenusal Number, say η_{x+1}, as a function of its antecedent, η_x, there will be another portion, consisting of terms with zero and negative indices of η_x having functions of x for their coefficients, which observation is incompetent to reveal, and with the nature of which we are at present unacquainted. The study of HAMILTON'S Numbers, far from being exhausted, has, in leaving our hands, little more than reached its first stage, and it is believed will furnish a plentiful aftermath to those who may feel hereafter inclined to pursue to the end the thorny path we have here contented ourselves with indicating, which lies so remote from the beaten track of research, and offers an example and suggestion of infinite series (as far as we are aware) wholly unlike any which have previously engaged the attention of mathematicians.

J. J. S. and J. H.

* Agreeing closely with what had been previously found by observation in § 1 (p. 297).

NOTE 1, page 299.—(September 17, 1887.)

It is easy to see that, if δM and $\delta\alpha$ are corresponding errors in the values of M and α respectively,

$$\delta M = (M \log_e M \log_e 2)\,\delta\alpha = (\cdot 38822\ldots)\,\delta\alpha$$

(since $M = 1\cdot 46544\ldots$, $\log_e M = \cdot 38220\ldots$, and $\log_e 2 = \cdot 69314\ldots$).

Hence, $\delta\alpha$ being intermediate between ·0000003 and ·0000006,

δM lies between ·000000116 and ·000000233.

The value of M (the base of the Hamiltonian Numbers) is thus found to be 1·465443 . . . , correct to the last figure inclusive.—J. J. S.

NOTE 2, page 300.—(September 17, 1887.)

This equation may be obtained more simply from the *fundamental* formula of HAMILTON (middle of above note). It follows from the law of derivation there given that, if we write ${}^1F_n = (1-x)^{-1}F_n - x^n$, and, in general, ${}^{j+1}F_n = (1-x)^{-1}\,{}^jF_n - x^n$, then $F_{n+1} = {}^{a_n}F_n$; and, consequently,

$$F_{n+1} - (1-x)^{-a_n}F_n = -x^n\{1 + (1-x)^{-1} + (1-x)^{-2} + \ldots + (1-x)^{-a_n+1}\}$$
$$= x^{n-1}\{(1-x) - (1-x)^{-a_n+1}\}.\text{—J. J. S.}$$

NOTE 3, page 303.—(September 19, 1887.)

It is curious to notice the sort of affinity which exists between a form of writing the scale of relation for BERNOULLI'S Numbers and that given at p. 289 for HAMILTON'S.

If we write $G_0 = 1$, $G_1 = -1$, $G_2 = (-4)B_1$, $G_3 = 0$, $G_4 = (-4)^2 B_2$, $G_5 = 0$, $G_6 = (-4)^3 B_3, \ldots$ then, using β_κ in the same sense as at p. 289, we shall find the scale of relation between the B's (BERNOULLI'S Numbers) is given by the equation

$$\sum_{\kappa=0}^{\kappa=i} (-)^\kappa \beta_\kappa i\,.\,G_{i-\kappa} = 0, \quad \textit{provided } i \textit{ is odd.}$$

On striking out the i which intervenes between β_κ and $G_{i-\kappa}$, so as to make the former operate on the latter, the equation becomes that given at p. 289 for the E's, the sharpened numbers of HAMILTON.—J. J. S.